Week 11: System Design and Polar Diagrams

Mahmut Selman Sakar

Institute of Mechanical Engineering, EPFL

Bode Form of the Transfer Function

- Components of transfer functions

  1. $constants$ (gain)
  2. $(j\omega)^n$
  3. $(j\omega\tau + 1)^{\pm 1}$
  4. $\left[\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1\right]^{\pm 1}$
  5. $e^{-ja\omega}$ (delay)

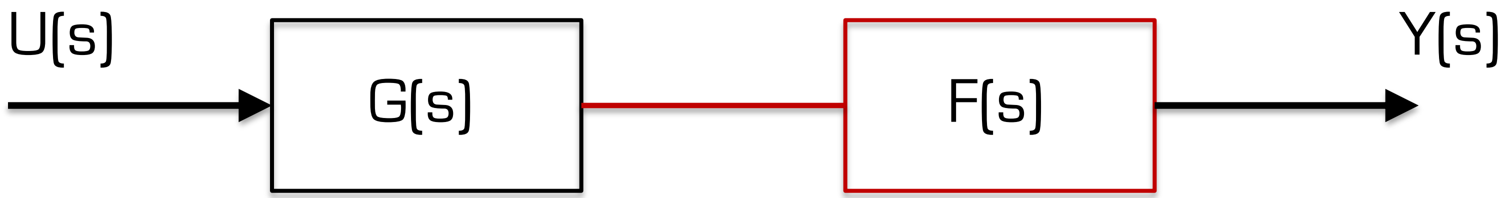
- Break points (corner frequency)

  2. $\omega_b = 1/\tau$
  3. $\omega_b = \omega_0$

- Bandwidth and Cut-off frequency

Filter Design

- The objective is to modify certain characteristics of system response
  - Magnitude and phase at a certain frequency
  - **Low-pass filter:** cut unwanted high-frequency signals
  - **High-pass filter:** cut unwanted low-frequency components
  - **Band-pass** and **notch filter**: attenuate specific frequencies
  - **All-pass filter** (phasor effect): only change phase



Example

Consider a mechanical system described by the following differential equation. The system is initially at rest.

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 2u(t)$$

a) Find the transfer function $G(s)$ of the system and sketch the Bode plot.

b) We would like to design a first order filter $F(s) = \dfrac{K}{\tau s + 1}$ in a way that the new system with the transfer function $G'(s) = G(s) \times F(s)$ has magnitude $|G'(j\omega)| = 1$ and phase angle $\phi = -3\pi/4$ at frequency $\omega = 1$.

Example

a) The transfer function can be calculated as:

$$G(s) = \frac{2}{s^2 + s + 1}$$

The second order term has a natural frequency of $\omega_0 = 1\text{rad/sec}$, the damping ratio is $\zeta = 0.5$, and the gain is 2. The resonance frequency is $\omega_r = \omega_0\sqrt{1-2\zeta^2} = 0.707$. The resonant peak is $R(\omega_r) = \dfrac{2}{2\zeta\sqrt{1-2\zeta^2}} = 2.31$.

Example

b) The filtered system is given by:

$$G'(s) = \frac{K}{\tau s + 1}\frac{2}{s^2 + s + 1}$$

The magnitude of the sinusoidal transfer function at $\omega = 1$ must be 1.

$$|G'(j\omega)| = \frac{2K}{\sqrt{\tau^2\omega^2 + 1}\sqrt{(1-\omega^2)^2 + \omega^2}} \rightarrow |G(\omega = 1)| = \frac{2K}{\sqrt{1+\tau^2}} = 1$$

And the phase angle of the sinusoidal transfer function at $\omega = 1$ must be $-3\pi/4$.

$$\phi(\omega = 1) = -\pi/2 - arctan(\tau) = -3\pi/4$$

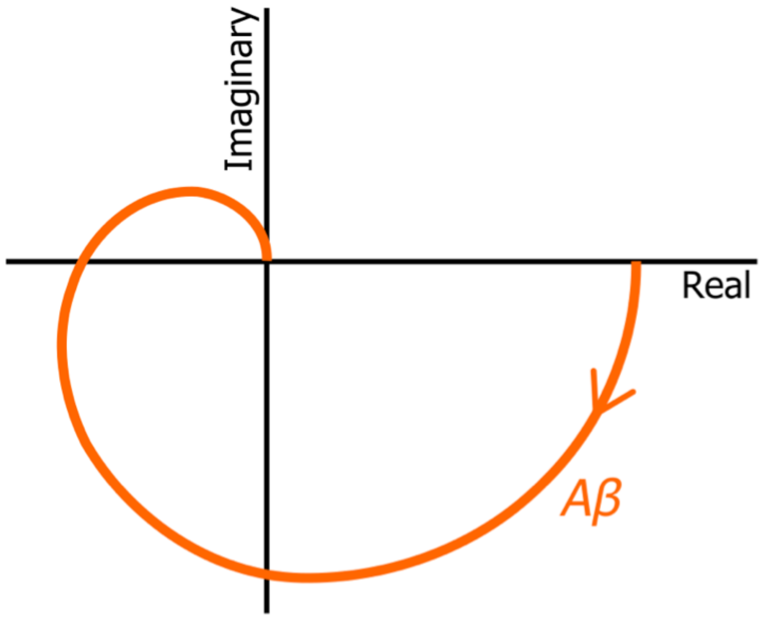
As a result, $\tau = 1$ and $K = \sqrt{2}/2 = 0.707$.

Nyquist Plot (or Nyquist Diagram)

- Commonly used to assess the stability of a system
  - Nyquist stability criterion

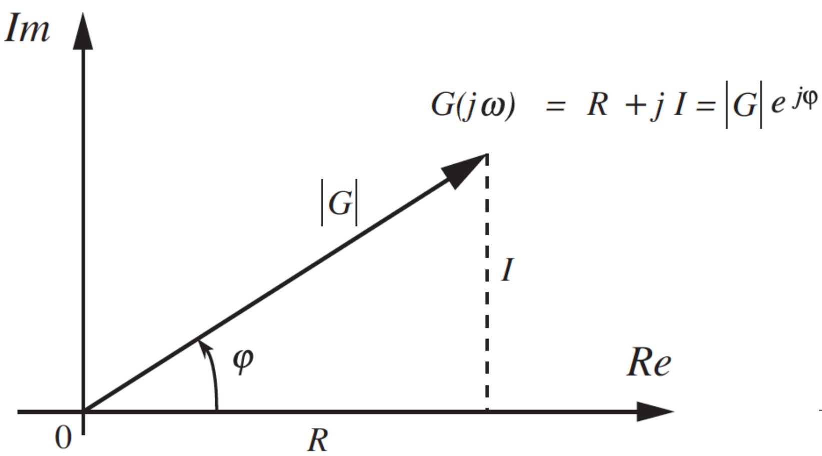
- In Cartesian coordinates, the real part of the transfer function is plotted on the X axis, and the imaginary part is plotted on the Y axis.

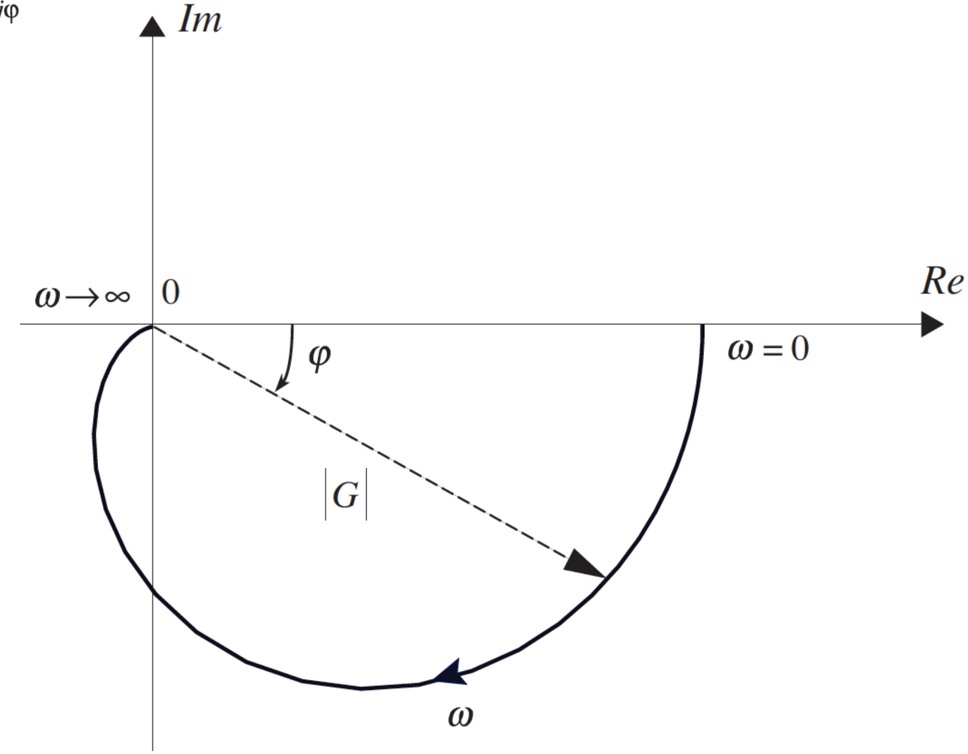
- The frequency is swept as a parameter, resulting in a plot based on frequency.



Nyquist Plot

- Magnitude and phase angle on the same graph
- Polar representation (Argand Diagram)





Nyquist Plot

- The frequency response (sinusoidal transfer function) G(jw) is plotted on the complex plane as a function of w
- It is convenient to sketch a Bode plot first, so that we can have a good idea of what the polar plot looks like

- Important considerations
  - Where does the plot intersect with the **unit circle**

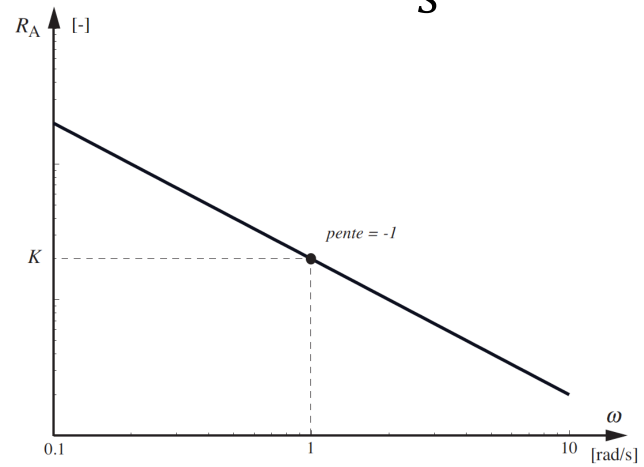
$$|G(j\omega)| = 1$$

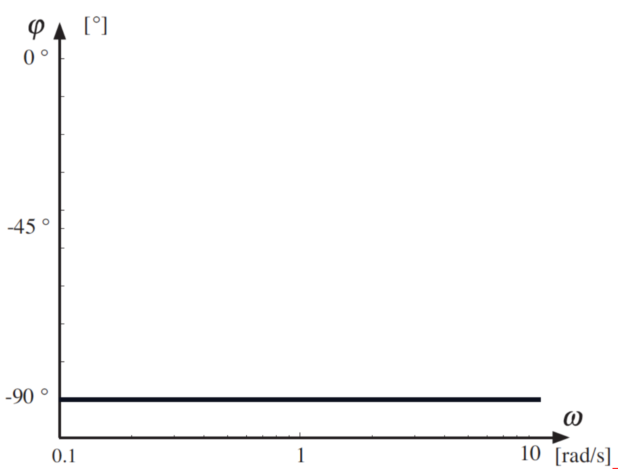
  - Where does the plot cross the **real axis**

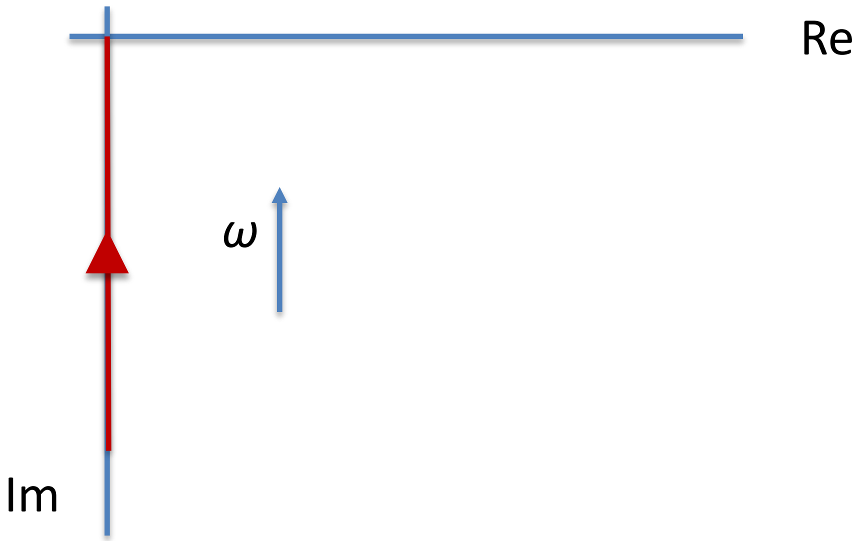
$$arg(G(j\omega)) = n \cdot 180°$$

Examples

$$G(s) = \frac{1}{s} \qquad G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega}$$



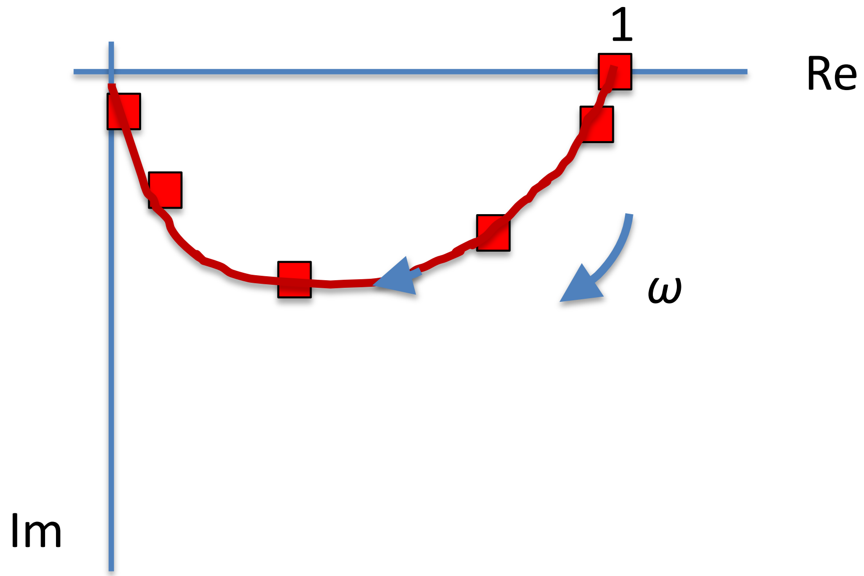


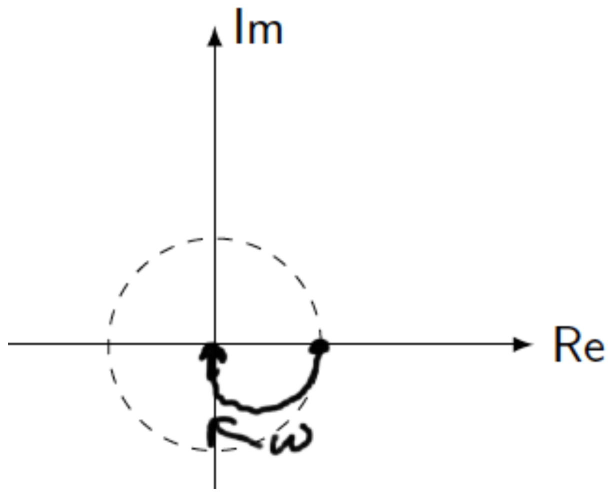


Examples

$$G(s) = \frac{1}{s+1} \qquad G(j\omega) = \frac{1}{j\omega+1} = \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2}$$

Frequency	0.1	0.5	1	2	10
G(jw)	0.99-j0.1	0.8-j0.4	0.5-j0.5	0.2-j0.4	0.01-j0.1

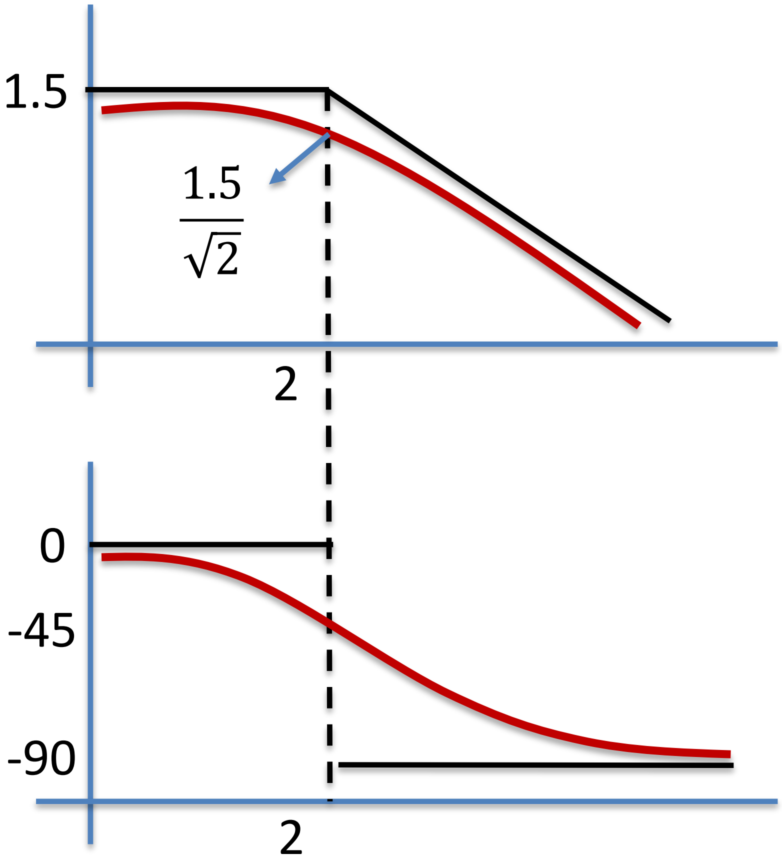


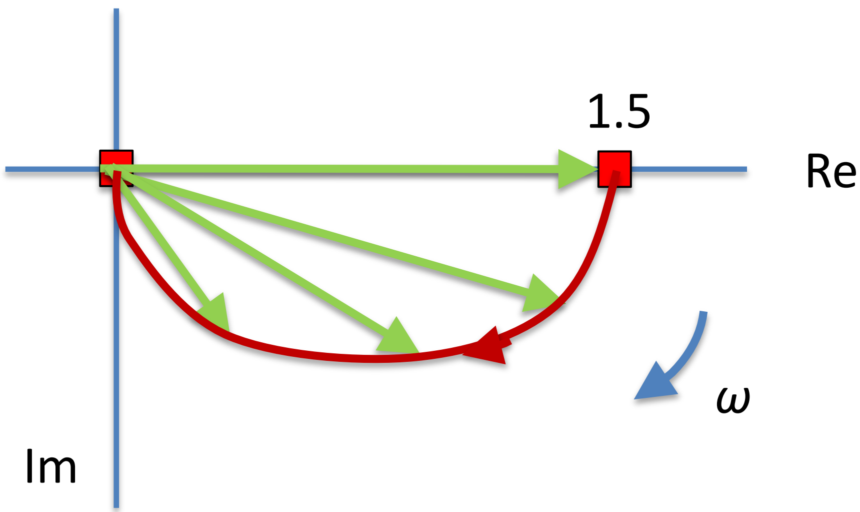


Examples

$$G(s) = \frac{3}{(s+2)}$$

- First sketch the Bode plot
- Mark the magnitude while following phase angle





Guidelines for quick sketches

- We want to sketch Nyquist diagram from the Bode diagrams

- Phase is decreasing (or becoming negative)
    - Plot is moving clockwise

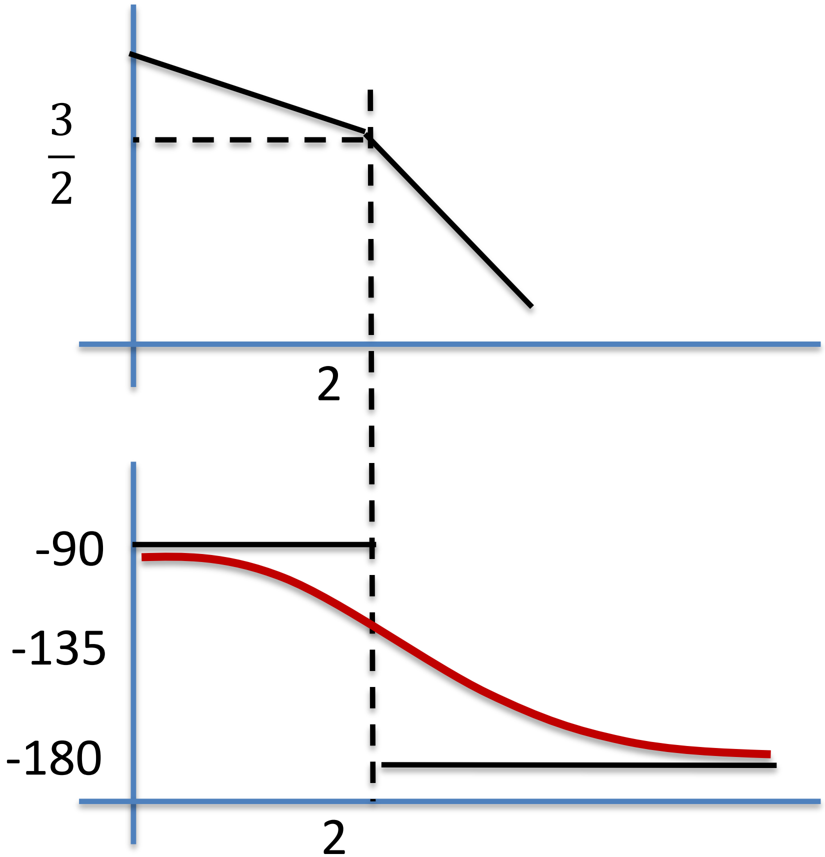
- Phase is increasing
    - Plot is moving counterclockwise

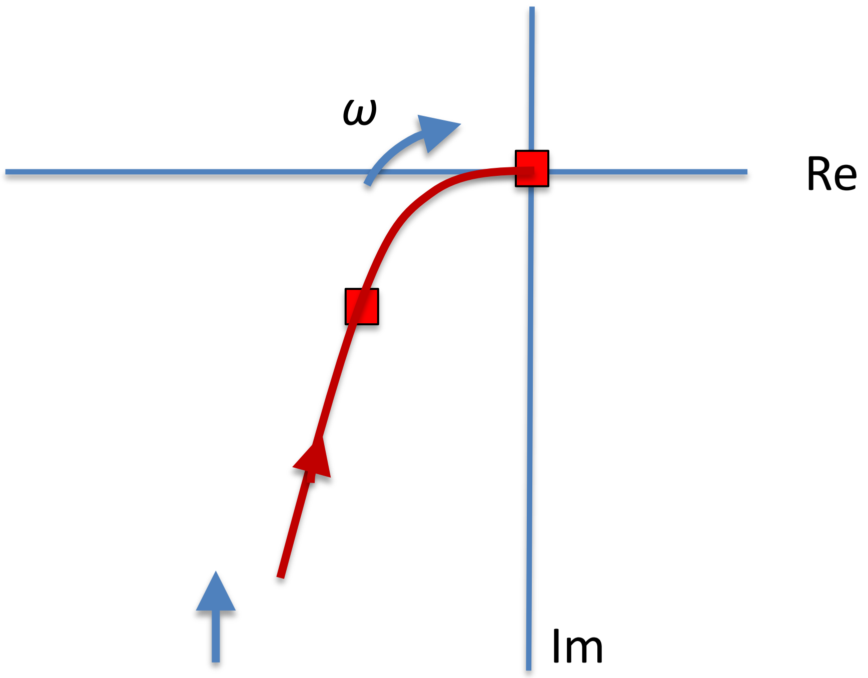
- Magnitude is decreasing
    - Plot is moving towards the origin

- Magnitude is increasing
    - Plot is moving away from the origin

Examples

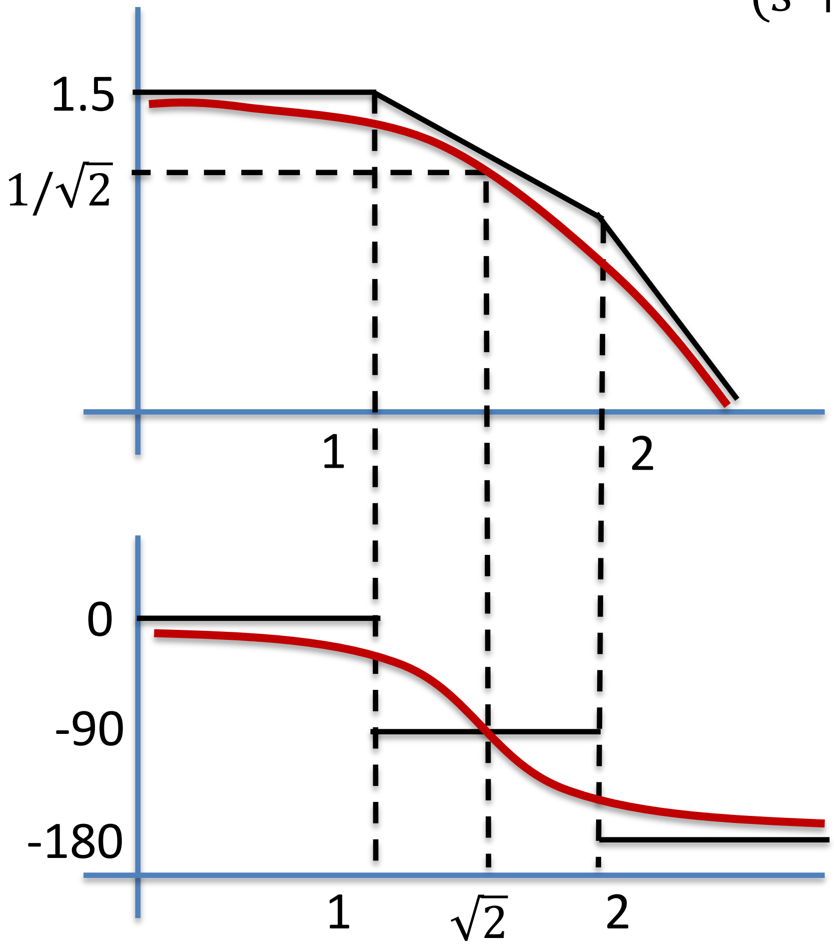
$$G(s) = \frac{3}{s(s+2)}$$

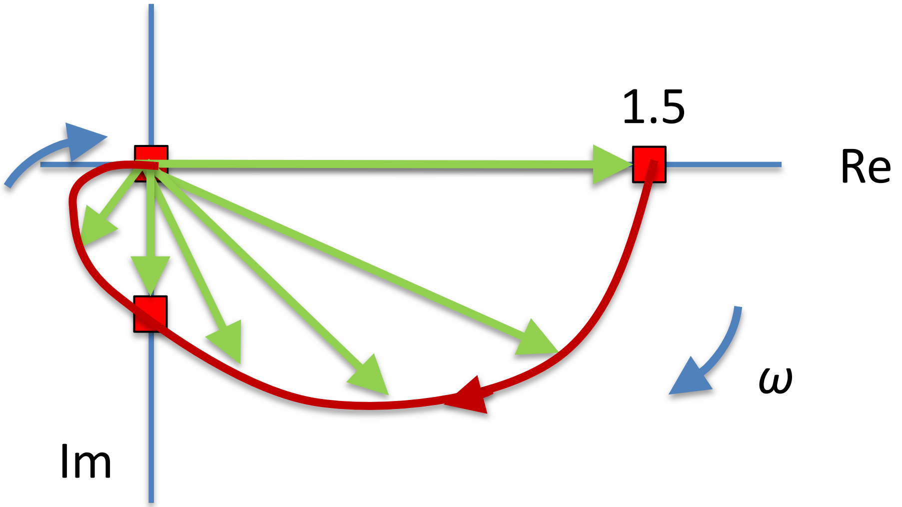




Examples

$$G(s) = \frac{3}{(s+1)(s+2)}$$

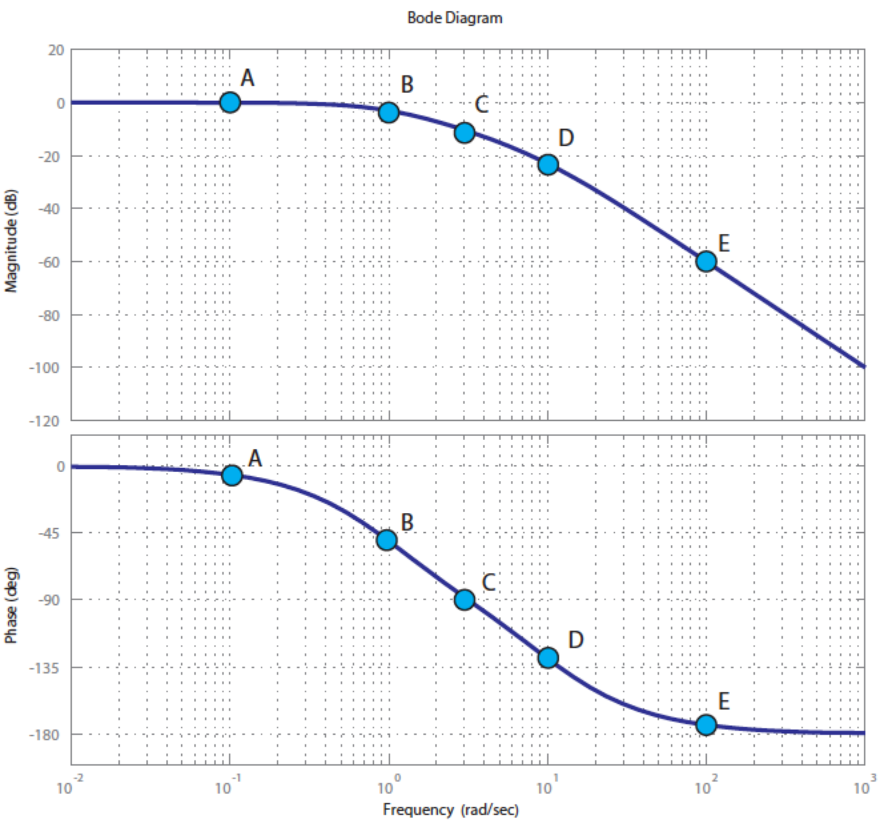


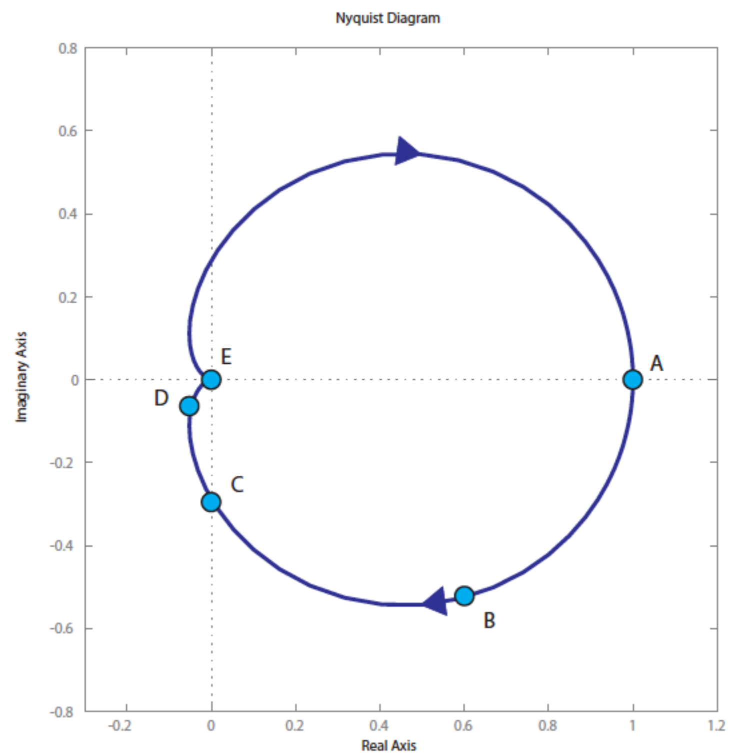


Examples

$$G(s) = \frac{1}{(s+1)(0.1s+1)}$$

Point	ω	$\angle G$	$\lvert G \rvert$
A	.1	$0°$	1
B	1	$-45°$	.7
C	3	$-90°$	.3
D	10	$-135°$	.07
E	100	$-175°$	.001

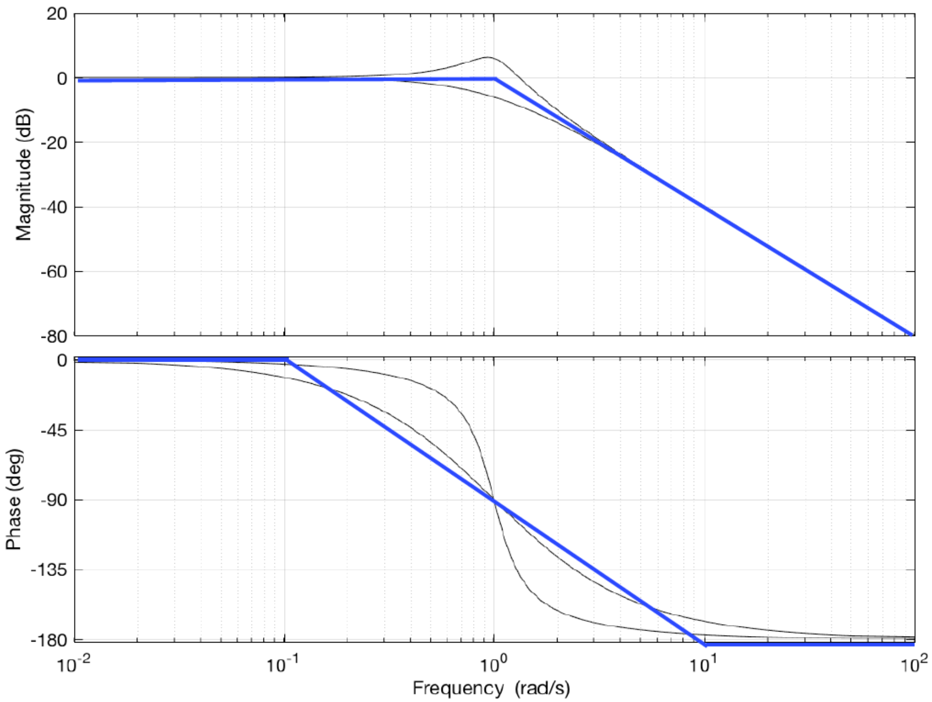


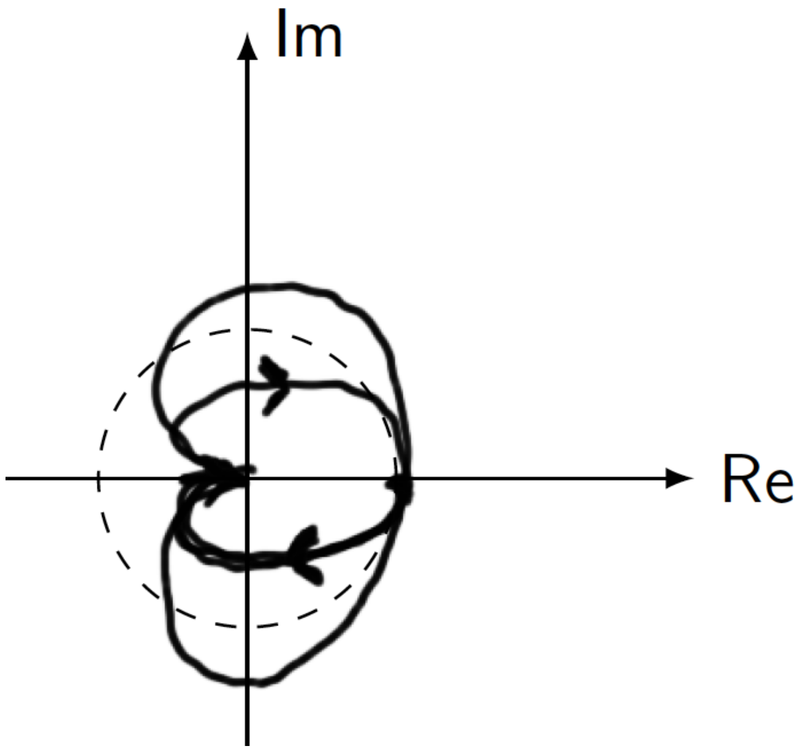


Examples

$$G_1(s) = \frac{1}{(s^2 + 0.5s + 1)}$$

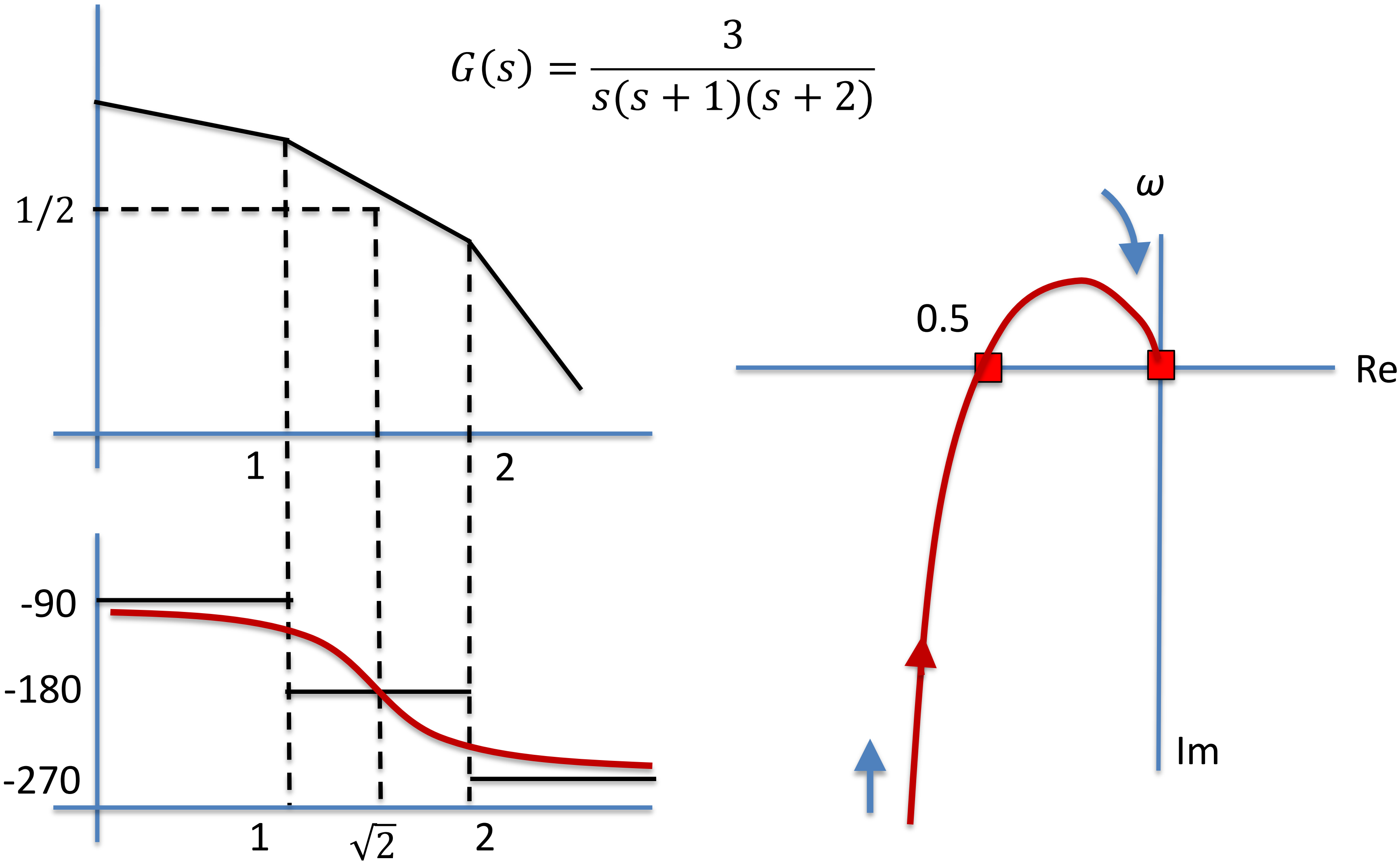
$$G_2(s) = \frac{1}{(s^2 + 2s + 1)}$$





Examples

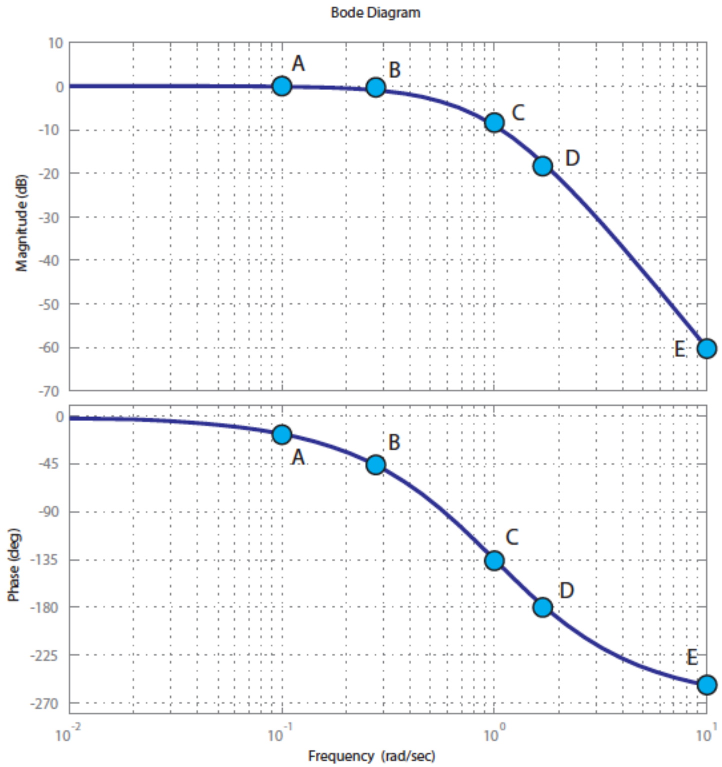
$$G(s) = \frac{3}{s(s+1)(s+2)}$$

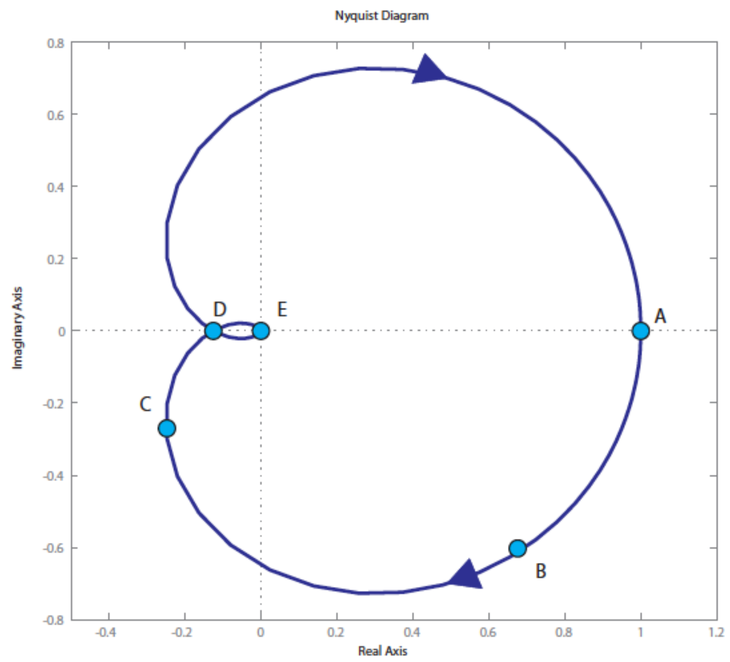


Examples

$$G(s) = \frac{1}{(s+1)^3}$$

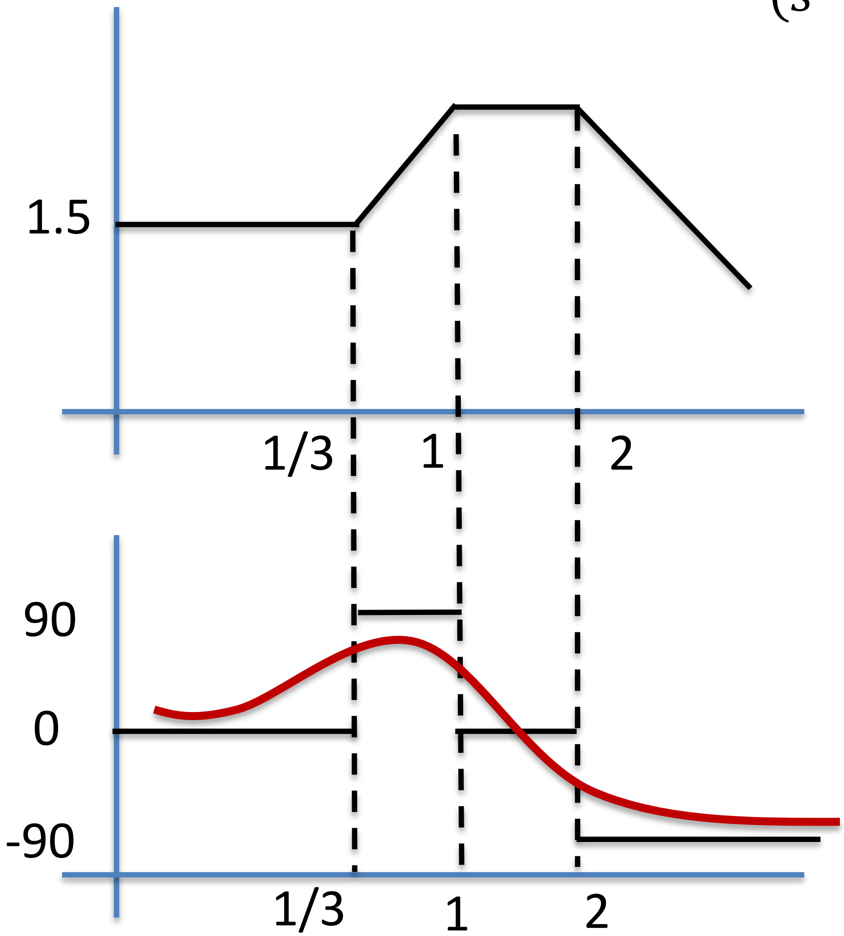
Point	ω	$\angle G$	$\|G\|$
A	.1	$0°$	1
B	.28	$-45°$	.95
C	1	$-135°$	.35
D	1.8	$-180°$	.1
E	10	$-260°$	.001

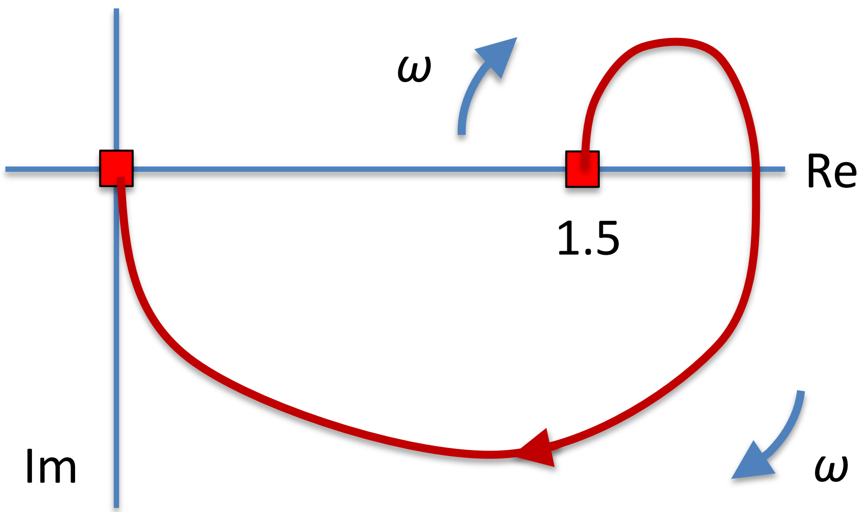




Examples

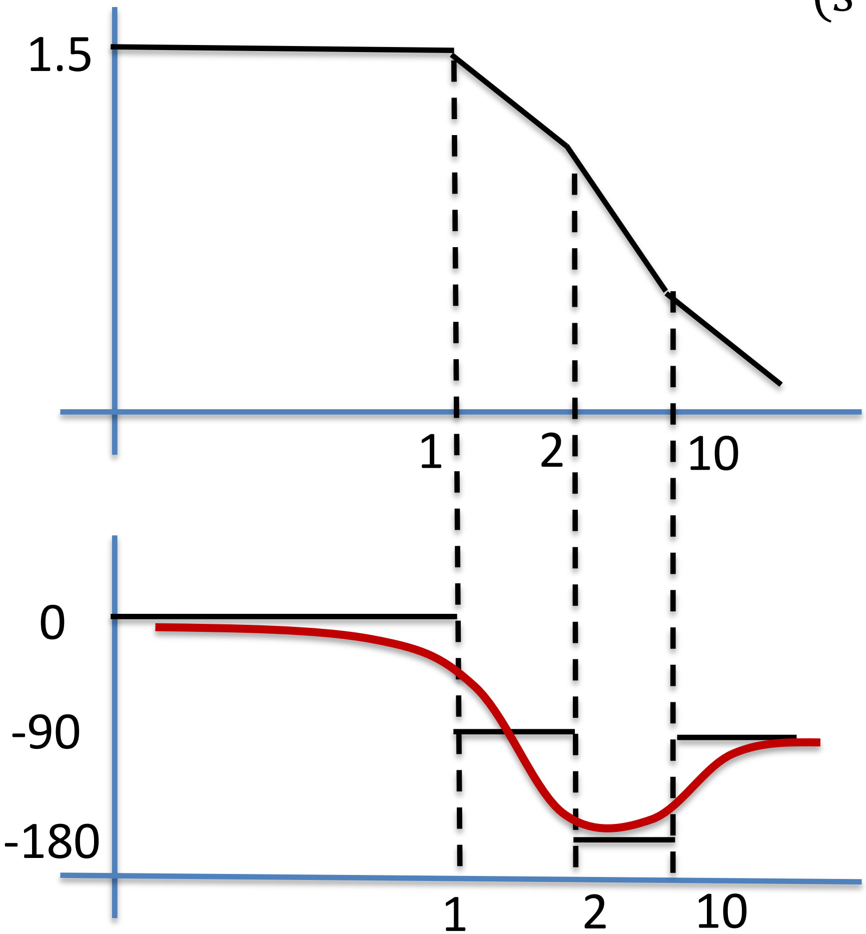
$$G(s) = \frac{9(s + 1/3)}{(s + 1)(s + 2)}$$

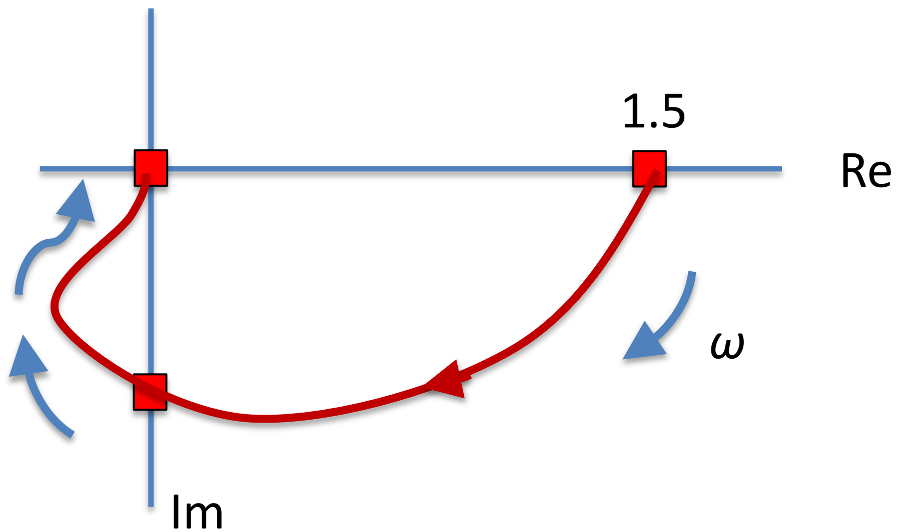




Examples

$$G(s) = \frac{0.3(s+10)}{(s+1)(s+2)}$$





Examples

- What if we have a time delay?
- Exponential term in the transfer function

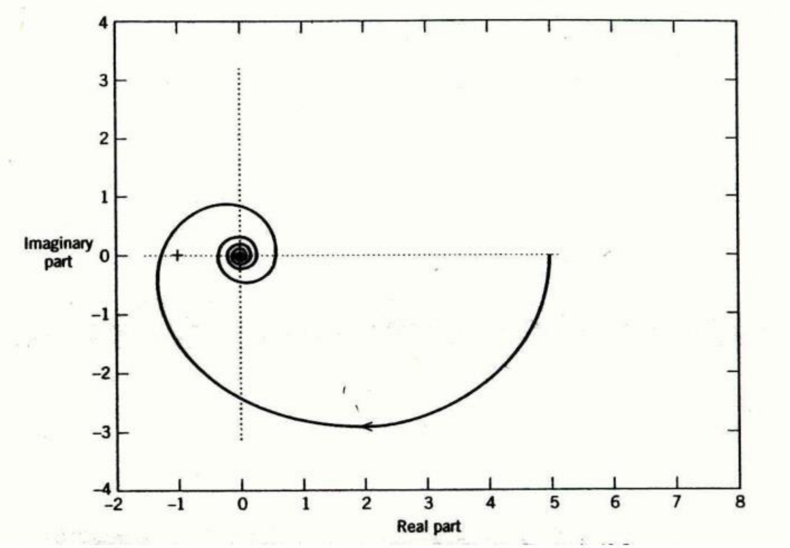


Figure 13.13 The Nyquist diagram for the transfer function in Example 13.5:

$$G(s) = \frac{5(8s+1)e^{-6s}}{(20s+1)(4s+1)}$$

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